

# Wave scattering by many small particles embedded in a medium.

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## Abstract

Theory of scattering by many small bodies is developed under various assumptions concerning the ratio  $\frac{a}{d}$ , where  $a$  is the characteristic dimension of a small body and  $d$  is the distance between neighboring bodies  $d = O(a^{\kappa_1})$ ,  $0 < \kappa_1 < 1$ . On the boundary  $S_m$  of every small body an impedance-type condition is assumed  $u_N = \zeta_m u$  on  $S_m$ ,  $1 \leq m \leq M$ ,  $\zeta_m = h_m a^{-\kappa}$ ,  $0 < \kappa$ ,  $h_m$  are constants independent of  $a$ . The behavior of the field in the region in which  $M = M(a) \gg 1$  small particles are embedded is studied as  $a \rightarrow 0$  and  $m(a) \rightarrow \infty$ . Formulas for the refraction coefficient of the limiting medium are derived under the assumptions: a)  $\kappa_1 = (2 - \kappa)/3$ ,  $0 < \kappa \leq 1$ , and b)  $\kappa_1 = 1/3$ ,  $\kappa > 1$ .

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## 1 Introduction

The theory of wave scattering by small bodies was originated by Rayleigh in 1871 [1]. In [4] this theory was developed for small bodies of arbitrary shapes, analytic formulas for the  $S$ -matrix for acoustic and electromagnetic

(EM) wave scattering by small bodies of arbitrary shapes have been derived. These formulas allow one to calculate the  $S$ -matrix with any desired accuracy. Analytic formulas for the electric and magnetic polarizability sensors have been derived for bodies of arbitrary shapes [2], [4]. In [3] – [11] a theory of wave scattering by many small bodies embedded in a bounded domain filled in by a material with known properties was developed. It was assumed in [9] and [10] that the characteristic size of the small particles (bodies) is  $a$ , that the distance  $d$  between two neighboring particles is of the order  $d = O(a^{1/3})$ , that the total number of the embedded particles  $M = O(\frac{1}{a})$ , and that the boundary condition on the boundary  $S_m$  of  $m$ -th particle  $D_m$  is of impedance type:

$$u_N = \zeta_M u \quad \text{on } S_m, \quad 1 \leq m \leq M, \quad (1)$$

where  $N$  is the unit normal to  $S_m$  directed out of  $D_m$ , and  $\zeta_m = \frac{h_m}{a}$ , where  $h_m$ ,  $\text{Im } h_m \leq 0$ , is a constant independent of  $a$ .

The waves in the original material are described by the equation

$$L_0 u_0 := [\nabla^2 + k^2 n_0^2(x)] u_0 = 0 \quad \text{in } \mathbb{R}^3, \quad (2)$$

where

$$n_0^2(x) = 1 \quad \text{in } D' = \mathbb{R}^3 \setminus D, \quad (3)$$

$D$  is a bounded domain, and  $n_0^2(x)$  is continuous in  $D$  (or piecewise-continuous with a finite number of discontinuities, which are smooth surfaces),  $\text{Im } n_0^2 \geq 0$ . The scattering solution to (2) satisfies the radiation condition

$$u_0 = e^{ik\alpha \cdot x} + v_0, \quad (4)$$

$$\frac{\partial v_0}{\partial r} - ikv_0 = o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty. \quad (5)$$

If small particles are embedded in  $D$ , then the scattering problem consists of finding the solution to the following problem:

$$L_0 u_M = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m, \quad (6)$$

$$\frac{\partial u_M}{\partial N} = \zeta_m u_M \quad \text{on } S_m, \quad 1 \leq m \leq M, \quad (7)$$

$$u_M = u_0 + v_M, \quad (8)$$

where  $u_0$  solves problem (2), (4), (5) and  $v_M$  satisfies the radiation condition similar to (5).

It is proved in [10] that problem (6) – (8) has a unique solution and this solution is of the form

$$u_M = u_0(x) + \sum_{m=1}^M \int_{S_m} G(x, t) \sigma_m(t) dt, \quad (9)$$

where  $G(x, y)$  is the Green function of the operator  $L_0$  for  $M = 0$ , i.e., in the absence of small particles:

$$L_0 G(x, y) = -\delta(x - y) \quad \text{in } \mathbb{R}^3, \quad (10)$$

$G$  satisfies the radiation condition (5), and  $\sigma_m$  solves the equation

$$u_{e_N} - \zeta_m u_e + \frac{A_m \sigma_m - \sigma_m}{2} - \zeta_m T_m \sigma_m = 0 \quad \text{on } S_m. \quad (11)$$

Here  $u_e$  is the effective field acting on the  $m$ -th particle:

$$u_e(x) := u_e^{(m)}(x) := u_M(x) - \int_{S_m} G(x, t) \sigma(t) dt, \quad x \in \mathbb{R}^3, \quad (12)$$

$$A_m \sigma_m := 2 \int_{S_m} \frac{\partial G(s, t)}{\partial N_s} \sigma_m(t) dt, \quad T_m \sigma_m := \int_{S_m} G(s, t) \sigma_m(t) dt. \quad (13)$$

It was proved in [10] that

$$G(x, y) = \frac{1}{4\pi|x - y|} [1 + O(|x - y|)], \quad |x - y| \rightarrow 0, \quad (14)$$

and one can differentiate formula (14).

The following result is also proved in [10]. Assume that  $D_m$  is a ball of radius  $a$  centered at a point  $x_m$ . Let  $h(x)$  be an arbitrary continuous function in  $D$ ,  $\text{Im } h(x) \leq 0$ ,  $\Delta_p \subseteq D$  be any subdomain of  $D$ , and  $\mathcal{N}(\Delta_p)$  be the number of particles in  $\Delta_p$ . Assume that

$$\mathcal{N}(\Delta_p) = \frac{1}{a} \int_{\Delta_p} N(x) dx [1 + o(1)], \quad a \rightarrow 0, \quad (15)$$

where  $N(x) \geq 0$  is a given continuous function in  $D$ . Let

$$p(x) := \frac{4\pi N(x) h(x)}{1 + h(x)}. \quad (16)$$

Finally, assume that  $\zeta_m := \frac{h(x_m)}{a}$ . Now the result can be formulated:

**Theorem 1** ([10]). *Under the above assumptions there exists the limit*

$$\lim_{a \rightarrow 0} \|u_e(x) - u(x)\|_{C(D)} = 0. \quad (17)$$

*The function  $u(x)$  solves the problem*

$$Lu := [\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in } \mathbb{R}^3, \quad (18)$$

$$u = u_0 + v, \quad (19)$$

*where  $u_0$  satisfies equations (2), (4), (5), the function  $v$  satisfies the radiation condition similar to (5), and*

$$q(x) := q_0(x) + p(x), \quad n^2(x) := 1 - k^{-2}q(x), \quad (20)$$

*where  $p(x)$  is defined in (16),*

$$q_0(x) := k^2 - k^2 n_0^2(x), \quad (21)$$

*and  $n_0^2(x)$  is the coefficient in (2).*

*The aim of this paper is to investigate the behavior of  $u_e(x)$  when the assumptions  $\zeta_m = \frac{h(x_m)}{a}$ ,  $d = O(a^{1/3})$ ,  $M = O(\frac{1}{a})$  are replaced by the following more general assumptions:*

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \quad d = O(a^{\kappa_1}), \quad M = O\left(\frac{1}{a^{3\kappa_1}}\right), \quad (22)$$

*where  $\kappa > -1$  and  $0 \leq \kappa_1 < 1$  are parameters.*

If  $\kappa_1 = 1$ , then the distance between neighboring particles is of the order of the size of a small particle. This is a special case which is not covered by a rigorous theory. However, if  $\kappa_1$  is close to 1, then practically the distance between neighboring particles is very close to the order  $a$  of the size of a small particle.

In [10] the theory was developed in detail in the case  $\kappa = 1$ ,  $\kappa_1 = \frac{1}{3}$ .

The questions we are interested in this paper are:

- 1) For what ranges of  $\kappa$  and  $\kappa_1$  the limit  $u(x)$  of  $u_e(x)$ , as  $a \rightarrow 0$ , does exist?
- 2) What is the equation which this limit  $u(x)$  solves?

The answers we give are:

1) If  $\kappa < 1$  and  $\kappa_1 = \frac{2-\kappa}{3}$ , then the limit (17) exists,

$$\sigma_m = -\frac{h(x_m)u_e(x_m)}{a^\kappa} (1 + o(1)), \quad Q_m = -4\pi h(x_m)a^{2-\kappa}u_e(x_m)$$

and the limiting function  $u(x)$  solves the following equation:

$$u(x) = u_0(x) - 4\pi \int_D G(x, y)h(y)N(y)u(y)dy. \quad (23)$$

Therefore,  $u$  solves equation (18) with  $q(x)$  given by (20),  $q_0(x)$  given by (21), and

$$p(x) = 4\pi h(x)N(x), \quad (24)$$

where  $N(x) \geq 0$  is defined by the formula:

$$\mathcal{N}(\Delta_p) = \frac{1}{a^{3\kappa_1}} \int_{\Delta_p} N(x)dx [1 + o(1)], \quad a \rightarrow 0. \quad (25)$$

2) If  $\kappa > 1$  then, as  $a \rightarrow 0$ ,

$$Q_m = -4\pi u_e(x_m)a(1 + o(1)), \quad \sigma_m = -\frac{u_e(x_m)}{a} (1 + o(1)),$$

and the limit (17) exists if  $\kappa_1 = \frac{1}{3}$ . The limiting function  $u(x)$  solves equation (23) with  $h(y) = 1$  and  $N(x)$  defined by (25). The function  $u(x)$  also solves equation (18) with  $q(x)$  given by (20),  $q_0(x)$  given by (21), and  $p(x)$  given by (24) with  $h(x) = 1$ .

In both cases,  $0 < \kappa < 1$  and  $\kappa > 1$ , we have  $\kappa_1 < 2/3$ . This implies that the total volume of the embedded particles tends to zero as  $a \rightarrow 0$ . Indeed, the order of the total number of the embedded particles is  $O(a^{-3\kappa_1})$ , and the total volume of the embedded particles is of the order  $O(a^{3-3\kappa_1}) \rightarrow 0$  as  $a \rightarrow 0$ .

Let us make a remark about the case when  $\kappa_1 = \frac{2-\kappa}{3} = 1$ . In this case  $\kappa = -1$  and  $\zeta_m = h(x_m)a$ . Moreover, one has:

$$Q_m \sim a^3 \left[ \frac{4\pi}{3} \Delta u_e(x_m) - 4\pi h(x_m) u_e(x_m) \right],$$

where  $\Delta = \nabla^2$  is the Laplacean, and

$$\sigma_m \sim u_{e_N} - h a u_e(x_m).$$

The quantity  $I_m := |G(x, y_m) Q_m| = O(a^{3-\kappa_1})$ , as  $a \rightarrow 0$ , and

$$J_m = \left| \int_{S_m} [G(x, t) - G(x, x_m)] \sigma_m(t) dt \right| = O(a^{2-2\kappa_1} a^2), \quad a \rightarrow 0.$$

For the relation  $J_m \ll I_m$  to hold as  $a \rightarrow 0$ , it is sufficient that the relation

$$a^{4-2\kappa_1} \ll a^{3-\kappa_1}$$

holds. For this relation to hold it is sufficient to have  $\kappa_1 < 1$ .

The relation  $J_m \ll I_m$  allows us to use formula (35), see below, i.e., approximate the exact formula (31) by an approximate formula (35) with an error which tends to zero as  $a \rightarrow 0$ .

Assuming  $\kappa_1 < 1$ , one has

$$u_e(x) = u_0 + \sum_{m=1}^M G(x, y^{(p)}) \left[ \frac{4\pi}{3} \Delta u_e(y^{(p)}) - 4\pi h(y^{(p)}) u_e(y^{(p)}) \right] a^3 \mathcal{N}(\Delta_p). \quad (26)$$

We have:

$$a^3 \mathcal{N}(\Delta_p) = \frac{a^3}{a^{3\kappa_1}} \int_{\Delta_p} N(x) dx [1 + o(1)] \approx a^{3-3\kappa_1} N(y^{(p)}) |\Delta_p|, \quad (27)$$

where  $o(1)$  tends to zero as  $a \rightarrow 0$ . For the limit of the sum in (26) to exist as  $a \rightarrow 0$ , it is necessary and sufficient that  $3 = 3\kappa_1$ , i.e.,  $\kappa_1 = 1$ . If  $\kappa_1 = 1$ , then the limit of  $u_e(x)$ , as  $a \rightarrow 0$  and  $\max_p \text{diam } \Delta_p \rightarrow 0$ , is the function  $u(x)$ , which solves the equation

$$u(x) = u_0(x) + \int_D G(x, y) \left[ \frac{4\pi}{3} \Delta u(y) - 4\pi h(y) u(y) \right] N(y) dy. \quad (28)$$

Applying operator  $L_0 = \nabla^2 + k^2 - q_0(x)$  to (28) and using equation (10), one gets

$$L_0 u = - \left[ \frac{4\pi}{3} \Delta u - 4\pi h(x) u(x) \right] N(x). \quad (29)$$

Thus

$$\left[ 1 + \frac{4\pi}{3} N(x) \right] \nabla^2 u + k^2 u - q_0(x) u - 4\pi h(x) N(x) u(x) = 0. \quad (30)$$

The solution  $u(x)$  to equations (18) or (30) is a locally  $H_{loc}^2$  function, where  $H_{loc}^2$  is the Sobolev space of twice differentiable in  $L^2$ -sense functions on every bounded open subset of  $\mathbb{R}^3$ . This local smoothness:  $u \in H_{loc}^2(\mathbb{R}^3)$ , follows from known results on elliptic regularity, provided that the coefficients  $q_0(x)$  and  $h(x)N(x)$  are in  $L_{loc}^2$ . If these coefficients are smoother, then  $u$  is smoother.

The assumption  $\kappa_1 < 1$  allows us to prove that formula (35) of Section 2 is a good approximation of  $u$  as  $a \rightarrow 0$ .

The conclusions, obtained under the assumption  $\kappa_1 = 1$  are not proven to be exact in the limit  $a \rightarrow 0$ .

In Section 2 we prove the results listed in the answers.

## 2 Proofs

In the proofs we use some arguments from [10].

Case 1). Consider first the case  $\kappa < 1$ . Let us write the exact formula (9) as follows:

$$u_M(x) = u_0(x) + \sum_{m=1}^M G(x, x_m)Q_m + \sum_{m=1}^M \int_{S_m} G(x, t)\sigma_m(t)dt, \quad (31)$$

where  $x_m$  is the center of the ball  $D_m$  and

$$Q_m := \int_{S_m} \sigma_m(t)dt. \quad (32)$$

One has the following estimates (see [10]):

$$|G(x, y)| \leq \frac{c}{|x - y|}, \quad |\nabla G(x, y)| \leq c \max\left(\frac{k}{|x - y|}, \frac{1}{(x - y)^2}\right), \quad (33)$$

where  $c > 0$  stands for various constants independent of  $a$ .

Let us estimate  $Q_m$ . Eventually we want to derive sufficient condition for the relation

$$I_m := |G(x, x_m)Q_m| \gg \left| \int_{S_m} [G(x, t) - G(x, x_m)]\sigma_m(t)dt \right| := J_m \quad (34)$$

to hold as  $a \rightarrow 0$  and  $|x - x_m| \gg a$ . This relation allows one to rewrite the exact formula (31) as an approximate formula:

$$u_M = u_0(x) + \sum_{m=0}^M G(x, y_m) Q_m, \quad |x - x_m| \gg a, \quad (35)$$

the error of which tends to zero as  $a \rightarrow 0$ .

To derive a formula for  $Q_m$ , integrate (11) over  $S_m$  and use the divergence theorem to get:

$$\frac{4}{3} \pi a^3 \Delta u_e(x_m) - \frac{h(x_m)}{a^\kappa} u_e(x_m) 4\pi a^2 = Q_m + \frac{h}{a^\kappa} \int_{S_m} dx \int_{S_m} \frac{\sigma_m(t)}{4\pi|s-t|}. \quad (36)$$

One has

$$\int_{S_m} ds \int_{S_m} \frac{\sigma_m(t) at}{4\pi|s-t|} = \int_{S_m} dt \sigma_m(t) \int_{S_m} \frac{ds}{4\pi|s-t|} = a Q_m. \quad (37)$$

Here we have used the formula

$$\int_{S_m := \{s: |s-x_m|=a\}} \frac{ds}{4\pi|s-t|} = a, \quad t \in S_m. \quad (38)$$

Thus, (36) yields:

$$Q_m = \frac{\frac{4}{3} \pi a^3 \Delta u_e(x_m) - 4\pi h(x_m) u_e(x_m) a^{2-\kappa}}{1 + h a^{1-\kappa}}. \quad (39)$$

If  $\kappa < 1$  and  $a \rightarrow 0$ , then (39) implies

$$Q_m = -4\pi h(x_m) u_e(x_m) a^{2-\kappa} [1 + o(1)], \quad a \rightarrow 0. \quad (40)$$

This is the formula for  $Q_m$  which we wanted to derive.

If  $a \ll 1$ , then a formula for  $\sigma_m$  can be derived as follows. The function  $u_e$  does not change at a small distance of order  $a$ . Therefore one can assume that in a neighborhood of  $D_m$  the function  $u_e$  is a constant, and one considers a static problem of finding  $\sigma_m$ :

$$u_M(x) = u_e(x_m) + \int_{S_m} \frac{\sigma_m(t) dt}{4\pi|x-t|}, \quad \frac{\partial u_M}{\partial N} = \zeta_m u_M \text{ on } S_m. \quad (41)$$



We look for the solution  $\sigma_m = ca^\gamma$ , where  $c$  and  $\gamma$  are constants. In this case we have:

$$\int_{S_m} \frac{\sigma_m(t)dt}{4\pi|s-t|} = ca^\gamma \frac{a^2}{|x-x_m|}, \quad |x-x_m| = O(a). \quad (42)$$

Using the impedance boundary condition (41) and choosing the origin at the point  $x_m$ , one gets

$$-ca^\gamma \frac{a^2}{a^2} = \frac{h(x_m)}{a^\kappa} [u_e(x_m) + ca^\gamma \frac{a^3}{a}].$$

From the above equation one derives:

$$ca^\gamma = -\frac{h(x_m)u_e(x_m)}{a^\kappa[a + h(x_m)a^{1-\kappa}]}. \quad (43)$$

If  $\kappa < 1$  and  $a \rightarrow 0$ , then equation (43) implies  $\gamma = -\kappa$  and  $c = -h(x_m)u_e(x_m)$ , so

$$\sigma_m = -\frac{h(x_m)u_e(x_m)}{a^\kappa} [1 + o(1)], \quad a \rightarrow 0. \quad (44)$$

This is the formula for  $\sigma_m$  which we wanted to derive.

Let us find sufficient conditions for the relation (34) to hold. Using estimates (33) and (39), we get

$$G(x, x_m)Q_m = O\left(\frac{a^{2-\kappa}}{a^{\kappa_1}}\right), \quad a \rightarrow 0, \quad |x-x_m| \geq d = O(a^{\kappa_1}). \quad (45)$$

Using (33) and (44) one gets:

$$J_m = O\left(\frac{a}{a^{2\kappa_1}} a^{2-\kappa}\right) = O(a^{3-\kappa-2\kappa_1}), \quad a \rightarrow 0, \quad |x-x_m| \geq d. \quad (46)$$

For (34) to hold it is sufficient to have:

$$a^{3-\kappa-2\kappa_1} \ll a^{2-\kappa-\kappa_1}, \quad a \rightarrow 0. \quad (47)$$

*This relation holds if  $\kappa_1 < 1$ .*

Thus, let us assume that  $\kappa < 1$  and  $\kappa_1 < 1$ , and use formulas (36), (39) and (40) to get

$$u_M(x) = u_0(x) - \sum_{m=1}^M G(x, x_m) 4\pi h(x_m)u_e(x_m)a^{2-\kappa}, \quad |x-x_m| \geq d. \quad (48)$$

Now we want to pass to the limit  $a \rightarrow 0$  in equation (48). To do this, let us partition the domain  $D$  into a union of small cubes  $\Delta_p$  centered at points  $y^{(p)}$  and having no common interior points. The side of  $\Delta_p$  is  $b \gg a$ . The number  $\mathcal{N}(\Delta_p)$  of small particles in  $\Delta_p$  by formula (25) is:

$$\mathcal{N}(\Delta_p) = \frac{1}{a^{3\kappa_1}} \int_{\Delta_p} N(x) dx [1 + o(1)] = \frac{1}{a^{3\kappa_1}} N(y^{(p)}) |\Delta_p| [1 + o(1)], \quad (49)$$

where  $o(1)$  in the second equation tends to zero as  $\text{diam } \Delta_p$  tends to zero, and  $|\Delta_p|$  is the volume of the cube  $\Delta_p$ . Write the sum in equation (48) as

$$\begin{aligned} & \sum_p G(x, y^{(p)}) 4\pi h(y^{(p)}) u_e(y^{(p)}) a^{2-\kappa} \sum_{x_m \in \Delta_p} 1 \\ &= \sum_p G(x, y^{(p)}) 4\pi h(y^{(p)}) u_e(y^{(p)}) \frac{a^{2-\kappa}}{a^{3\kappa_1}} N(y^{(p)}) |\Delta_p| (1 + o(1)). \end{aligned} \quad (50)$$

The sum in (50) is a Riemannian sum for the integral

$$\int_D G(x, y) 4\pi h(y) N(y) u_e(y) dy. \quad (51)$$

The limit of the sum in (50), as  $a \rightarrow 0$ , exists if and only if  $2 - \kappa = 3\kappa_1$ , i.e.,  $\kappa_1 = (2 - \kappa)/3$ . Note that if  $0 < \kappa < 1$ , then  $0 < \kappa_1 < 2/3$ .

In the region  $|x - x_m| \geq d$  one has:

$$|u_M(x) - u_e(x)| \leq O\left(\frac{a}{a^{\kappa_1}}\right) = o(1), \quad a \rightarrow 0. \quad (52)$$

The number of small particles in a unit cube is  $O(\frac{1}{d^3}) = O(\frac{1}{a^{3\kappa_1}})$  if  $d = O(a^{\kappa_1})$ , where  $d$  is the distance between two neighboring particles.

We assume that the functions  $h(y)$ ,  $u_e(y)$  and  $G(x, y)$  are continuous functions of  $y$ , so the error of replacing, for example,  $h(y_m)$  by  $h(y^{(p)})$ , where  $y_m \in \Delta_p$ , goes to zero as  $\text{diam } \Delta_p \rightarrow 0$ . The function  $G(x, y)$  is not continuous as  $y \rightarrow x$ , but  $G(x, y)$  is absolutely integrable, so one may remove a small neighborhood of the singular point  $x$  in the integral (51) and the change of this integral will be negligible if the neighborhood is sufficiently small. The function  $h$  is at our disposal, and we choose it to be continuous. The continuity of  $u_e$  and of its limit  $u$  follows from the relation  $u \in H_{loc}^2(\mathbb{R}^3)$ . A more detailed argument is given in [10].

Assuming that  $\kappa_1 = (2 - \kappa)/3$  and passing to the limit  $a \rightarrow 0$  in (48) yields equation (23). Applying to this equation the operator  $L_0$  and using equations (2) and (10), one gets equation (18) with  $q(x)$  given by (20) and  $p(x)$  given by (24).

We have proved all the claims in the answer to question 1).  $\square$

Note that if  $\kappa < 0$ , then the impedance parameter  $ha^{-\kappa}$  tends to zero as  $a \rightarrow 0$ .

Case 2). Let us justify the answer to question 2). We assume now that  $\kappa > 1$ . Then (39) implies

$$Q_m = -4\pi u_e(x_m)a[1 + o(1)], \quad a \rightarrow 0, \quad (53)$$

and equation (43) yields

$$ca^\gamma = -\frac{u_e(x_m)}{a}, \quad (54)$$

so

$$\gamma = -1, \quad c = -u_e(x_m),$$

and

$$\sigma_m = -\frac{u_e(x_m)}{a} [(1 + o(1))], \quad a \rightarrow 0. \quad (55)$$

Let us check when the relation (34) holds, i.e., when formula (35) is valid, in other words, when formula (35) yields an accurate approximation of  $u_M$ , defined by formula (31). From (33) and (53) we conclude, using the relation  $d = O(a^{\kappa_1})$ , that

$$|G(x, x_m)Q_m| = O\left(\frac{a}{d}\right) = O(a^{1-\kappa_1}), \quad a \rightarrow 0, \quad |x - x_m| \geq d. \quad (56)$$

Furthermore, using (55) and (33), one gets:

$$J \leq O\left(\frac{a}{a^{2\kappa_1}} a^{2-1}\right) = O(a^{2-2\kappa_1}), \quad a \rightarrow 0. \quad (57)$$

The relation (34) holds if  $a^{2-2\kappa_1} \ll a^{1-\kappa_1}$ , that is, if  $\kappa_1 < 1$ .

Let us assume that  $\kappa_1 < 1$ , so that formula (35) is applicable. We repeat the arguments given below formula (48). Due to formula (53), now formula (48) takes the form:

$$u_M(x) = u_0(x) - \sum_{m=1}^M G(x, x_m) 4\pi u_e(x_m) a. \quad (58)$$

We conclude from this formula that  $u_e(x)$  tends to the limit  $u(x)$ , and  $u$  solves the equation:

$$u(x) = u_0(x) - \int_D G(x, y) 4\pi N(y) u(y) dy, \quad (59)$$

provided that

$$\kappa_1 = \frac{1}{3}, \quad (60)$$

and  $N(x)$  is defined by the formula (25) for any subdomain  $\Delta \subset D$ , where  $\mathcal{N}(\Delta)$  is the number of small particles in  $\Delta$ .

Applying the operator  $L_0$  to (59) one gets equation (18) for  $u(x)$ , with  $q(x)$  given by (20),  $q_0(x)$  given by (21), and  $p(x)$  given by the formula

$$p(x) = 4\pi N(x). \quad (61)$$

Since  $N(x) \geq 0$ , the function  $p(x)$  is nonnegative.

The assumption  $\kappa > 1$  leads to the equation (18) with the potential  $q(x)$  which can vary much less than in the case  $\kappa \leq 1$ , because the function  $h(x)$  does not enter in the definition of  $q(x)$  when  $\kappa > 1$ .

### 3 Creating materials with a desired refraction coefficient

If  $\kappa < 1$  and  $\kappa_1 = (2 - \kappa)/3$ , then equations (18), (20) and (24) hold. Thus, given  $n_0^2(x)$  and  $n^2(x)$ , one calculates

$$p(x) = k^2[n_0^2(x) - n^2(x)] := p_1(x) + ip_2(x). \quad (62)$$

From (24) and (62) one gets an equation for finding  $h(x) := h_1(x) + ih_2(x)$  and  $N(x) \geq 0$ :

$$4\pi[h_1(x) + ih_2(x)]N(x) = p_1(x) + ip_2(x). \quad (63)$$

Thus

$$N(x)h_1(x) = \frac{p_1(x)}{4\pi}, \quad N(x)h_2(x) = \frac{p_2(x)}{4\pi}. \quad (64)$$

There are many solutions  $\{h_1, h_2, N\}$  of two equations (64) for the three unknown functions  $h_1, h_2, N(x)$ ,  $h_2 \leq 0$ ,  $N \geq 0$ . The condition  $\text{Im } n^2(x) \geq 0$  implies  $\text{Im } p = p_2 \leq 0$ , which agrees with the inequalities  $h_2 \leq 0$ ,  $N \geq 0$ . One takes  $N(x) = h_1(x) = h_2(x) = 0$  at the points at which  $p_1(x) = p_2(x) = 0$ . At the points at which  $|p(x)| > 0$ , one may take

$$N(x) = N = \text{const}, \quad h_1(x) = \frac{p_1(x)}{4\pi N}, \quad h_2(x) = \frac{p_2(x)}{4\pi N}. \quad (65)$$

Let us partition  $D$  into a union of small cubes  $\Delta_p$ , which have no common interior points, and which are centered at the points  $y^{(p)}$ , and embed in each cube  $\Delta_p$  the number

$$\mathcal{N}(\Delta_p) = \left[ \frac{1}{a^{2-\kappa}} \int_{\Delta_p} N(x) dx \right] \quad (66)$$

of small balls  $D_m$  of radius  $a$ , centered at the points  $x_m$ , where  $[c]$  stands for the integer nearest to  $c > 0$ . Let us put these balls at the distances  $O(a^{\frac{2-\kappa}{3}})$ , and prepare the boundary impedances of these balls equal to  $\frac{h(x_m)}{a^\kappa}$ .

*Then the resulting material, which is obtained by embedding small particles into  $D$  by the above recipe, will have the desired refraction coefficient  $n^2(x)$  with an error going to zero as  $a \rightarrow 0$ .*

## 4 Conclusions

Wave scattering by many small particles, embedded into a material with a known refraction coefficient, is studied in this Letter under various assumptions about the orders  $\kappa_1$  and  $-\kappa$  with respect to powers of  $a$  of the distances  $d = O(a^{\kappa_1})$  between the neighboring small particles and their boundary impedances  $\zeta = O(a^{-\kappa})$ .

If  $a$  is the characteristic size of a spherical small particle  $D_m$ ,  $d = O(a^{\kappa_1})$  is the distance between the neighboring particles,  $\zeta_m = \frac{h(x_m)}{a^\kappa}$  is the boundary impedance, and  $x_m$  is the center of  $D_m$ , then the equations are derived, as  $a \rightarrow 0$ , for the effective field in the medium, consisting of many small particles, embedded in a given material (according to the recipe, derived in Section 2), under the following assumptions:

- 1)  $0 < \kappa \leq 1$ ,  $\kappa_1 = \frac{2-\kappa}{3}$ , and (25) holds,
- 2)  $\kappa > 1$ ,  $\kappa_1 = \frac{1}{3}$ , and (25) holds.

A remark is made about the case  $\kappa_1 = 1$  and  $\kappa = -1$ . In this case  $d = O(a)$  is of the order of the size of a small particle.

The results are used for formulating a recipe for creating materials with a desired refraction coefficient. This recipe is similar to the one, given in [8], [9] in the case  $\kappa_1 = 1/3$  and  $\kappa = 1$ .

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